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IMPROVEMENT OF MATHEMATICAL MODELS OF TWO-STAGE RESOURCE ALLOCATION PROCESSES IN EMERGENCY LOGISTICS SYSTEMS

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Abstract. The paper addresses the problem of optimizing the placement of facilities and the distribution of material resources in the context of emergency logistics. The relevance of the research is driven by the increasing frequency of natural and man-made disasters, which necessitates effective response planning to mitigate losses and threats to human life. The focus is on the development of mathematical models and methods aimed at improving the efficiency of logistical operations.

A new approach to mathematical modeling of multi-stage logistics processes is proposed, which combines the ideas of optimal placement of intermediate distribution centers and multiplex territorial zoning. Existing models of facility location problems with two-stage resource distribution have been improved by assigning each end consumer to several nearest intermediate distribution centers, which increases the reliability of service delivery to those in need. The facilities to be located may serve either as primary collection points for the population in case of evacuation from an emergency area, or as distribution and supply points for essential goods delivered from state reserve warehouses or other hubs and then distributed among intermediate centers for delivery to residents of the affected region.

The mathematical formulation is based on the theory of continuous optimal partitioning problems, duality theory, linear programming methods of the transportation type, and modern algorithms of nonsmooth (derivative-free) optimization. A numerical scheme is proposed for solving optimal flow distribution problems in multi-level transport and logistics networks.

The results have practical significance for planning logistics operations in emergency situations, particularly for the effective location of medical and humanitarian aid points, resource allocation, and population evacuation. The proposed approach enables the implementation of a comprehensive decision support system for crisis response management and can be applied to a wide range of strategic problems in industrial, social, and economic domains, providing effective support for decision-making in complex logistics systems.

Keywords: optimization, transport and logistics system, multi-stage distribution, decomposition of a continuous set, system approach

1. Introduction

The relevance of improving existing and developing new mathematical models and optimization methods for facility location and distribution of material resources in emergency logistics is driven by the increasing number of natural and man-made disasters. Immediate response to emergencies is the key to mitigating these threats and losses. Since response time depends on the number and location of emergency service facilities, the problem of determining their optimal number and best location has strategic importance and attracts great interest from researchers [13].

The high uncertainty of events, dynamic environmental conditions, and other factors require a systematic approach to planning preventive measures and prompt response to emergency situations, involving methods of mathematical and computer modeling.

The object of research is multi-stage logistics processes in emergency situations. The subject of research is models and methods of optimal distribution of emergency logistics system units and distribution of material flows between them.

The purpose of the study is to improve the efficiency of logistics operations and

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emergency response through mathematical and computer modeling of rational distribution of transport and material resources in emergency logistics systems.

Mathematical models of two-stage evacuation processes are presented in [2]. These models consider the optimal distribution of human flows in a transport system with two-level units – first-stage centers (medical stations receiving citizens from affected areas) and second-stage centers (specialized emergency service units providing further assistance to the evacuated population). It was found that the mathematical description of such processes corresponds to the problems of optimal partitioning of continuous sets with the placement of subset centers and additional connections. [3]. The universality of this class of problems was demonstrated in [2] through its application not only to evacuation processes with assembly, intermediate, and reception points but also to processes related to delivering essential goods to the affected population. In such cases, it is necessary to determine the volume of material flows and logistics connections between state reserve warehouses, distribution centers, and final delivery points in disaster areas. In this study, the above models and their implementation methods are generalized to the case where residents of the affected area can be served by several nearest centers.

Many problems in emergency response planning are presented as classical optimization problems, including resource distribution, facility location, vehicle routing, and optimal control of emerging processes. A comprehensive review of scientific works published in the first 20 years of this century on humanitarian logistics and optimization problems related to evacuation planning, route finding, and shelter location during urban emergencies, intentional actions, or natural disasters is given in [1].

We present an overview of research over the past decade, focusing on existing approaches to modeling two-stage evacuation or distribution processes. For example, [4] introduces a two-stage model for reliable facility location when some facilities may be disrupted, e.g., by a natural disaster. Planning is carried out in two stages: a robust network of service centers and their assigned clients is designed in the proactive phase, while in the reactive phase, when a service center is destroyed, its clients are reassigned to other available facilities. This model is based on the classical p-center problem. In [5], the authors present the results of their 15-year research on modeling and developing solution methods for two-stage facility location problems without capacity constraints. The industrial application in freight transportation, which initially sparked the authors' interest in this problem, led to the study of its generalization to the case with modular costs.

A significant part of the mathematical description of two-stage location–allocation processes consist of stochastic models. In [6], a new class of two-stage stochastic facility location problems with unsecured facilities is introduced, taking into account the "nervousness" of the system that may arise due to decision-making under uncertainty. The authors attempt to address a practical issue: adaptive uncertainty-aware distribution decisions at the second stage may significantly deviate from the corresponding first-stage allocation decisions. To solve such models, exact scenario decomposition algorithms of the Benders type have been developed. The facility location problem in which customer demand is highly uncertain and information about the distribution of uncertainty is unavailable is considered in [7]. A two-stage stochastic mixed-integer programming problem is formulated here: optimal facility selection at the first stage and optimal operation decisions for each facility at the second stage.

In [8], two classes of multi-objective two-stage stochastic programs on finite probability spaces with multidimensional risk constraints are examined. The first-stage problem includes a multidimensional stochastic benchmarking constraint based on a vector random variable. The vector random represents multiple, possibly stochastic, conflicting performance indicators that are associated with the decisions of the other stage.

The study [9] develops a multi-objective stochastic programming model to enhance disaster preparedness and response, with particular focus on the critical first 72 hours after earthquakes. The aim is to optimize the distribution of resources, temporary medical centers, and medical personnel for effective life-saving efforts. Dynamic and stochastic programming, as well as discrete Markov chains, are employed here.

The work [10] is devoted to the optimization of resource distribution during natural disasters accompanied by the emergence of secondary hazards. A two-stage stochastic optimization model is proposed to simulate random occurrences of multiple disasters in order to optimize the distribution of rescue teams, warehouses, and medical resources.

In [11], the problem of locating multiproduct facilities in a two-stage supply chain is studied, including the determination of warehouse (distribution center) locations, their capacities, and optimization of product flows from plants to warehouses and customers. Plants are assumed to have limited production capacities, while potential warehouses offer several possible capacity levels to choose from. The authors introduce two Mixed-Integer Linear Programming models, aiming to minimize fixed warehouse opening costs and transportation costs. In the first model, warehouse capacity is defined by the maximum number of each product that can be stored, while in the second it is defined by warehouse size (volume).

In [12], a multi-period maximal covering location model for medical services is evaluated, which takes into account the inter-service dimension as well as equitable access. A two-stage optimization strategy is applied: at the first stage, facilities are located to maximize covered demand, while at the second stage patients are allocated to the maximum-coverage facilities over several time periods. The facility location problem with uncertain customer demand is studied in [13]. The authors propose a two-stage formulation of this problem with mixed-integer "wait-and-see" decisions and discrete uncertainty in the objective function.

One of the most widely used research approaches to solving the emergency facility location problem is its formulation as a discrete coverage-based emergency facility location model [1, 16]. In [1], a comprehensive review of this problem is presented, including mathematical models, their extensions, and applications. Solution methods and promising future research directions based on coverage models are also discussed. In [16], the problem of optimal placement of rescue facilities with minimum service time to the farthest client in a given region is examined. The mathematical models are discrete-continuous multiple-coverage problems of a bounded set in E_2 with minimum-radius circles. The centers of the circles forming the coverage are located on a finite set of candidate locations. The objective is to minimize service time even when the center closest to the affected resident is unable (for certain reasons) to provide the required assistance.

The article [14] presents a bi-objective model for determining the location of emergency logistics facilities, considering location costs, human resource planning, demand uncertainty, and road conditions.

A comprehensive methodological review of research related to the use of mathematical modeling, optimization, and machine learning methods in emergency response planning is provided in [15]. Machine learning offers an adaptive, data-driven approach to disaster response. The machine learning and reinforcement learning (RL) are particularly effective in problems requiring real-time adaptation compared to transportation optimization models. However, the implementation of machine learning in disaster response faces challenges such as data availability and quality, interpretability of learned models, and generalization of solutions to different disaster scenarios.

2. Methods

In this work, the object of study is the two-stage distribution of material flows in emergency logistics systems, whether for population evacuation from a disaster-affected area or delivery of essential goods from state reserve warehouses to residents through intermediate collection or distribution centers. As shown in [2], the mathematical framework describing such models is the same – optimal partitioning problems of continuous sets with additional constraints. Therefore, we will use the term "i-th stage centers" for the respective units of the logistics system. Unlike the models proposed in [2, 3], in this work, it is assumed that residents of the affected area may be served by several nearest centers. Thus, we generalize the developed mathematical models of two-stage location/distribution to the case of multiplex zoning of the territory [17, 18].

We formulate several assumptions regarding the input data. We assume that:

1. The logistics centers perform the functions of distributing emergency relief materials and planning human resources. The construction and maintenance costs of each intermediate logistics center are considered known and remain constant throughout the studied period.
2. The time required to move between demand points and centers, as well as between the two stages of centers, is directly proportional to the distance.
3. The cost of establishing and equipping the second-stage centers is assumed to be known for all units.
4. The distance from each logistics center to each demand point is considered known and does not change over time.
5. Each logistics center, whether it is a final collection center for the affected population or a state reserve warehouse, can accumulate or direct human or material resources from/to several intermediate points. Each intermediate distribution center is assigned a specific zone of the territory affected by the emergency for servicing. At the same time,

it is assumed that residents may use the services of any of the k nearest centers, which e ensures a flexible and reliable logistics chain.

6. The territory affected by the emergency is densely populated, and the distribution of residents, and thus the demand for emergency aid, can be described analytically by a certain function.

7. All resources accumulated in an intermediate distribution center must be delivered to the final recipient.

Thus, it provides for: 1) multiplex zoning of the territory, i.e. overlapping zones in case the nearest center is unable to provide the customer with the service; 2) two-stage resource distribution; 3) optimal placement of intermediate distribution centers. Let us now turn to the mathematical description of the above processes.

Let Ω be the territory that has been damaged (can be damaged) as a result of an emergency event, m^2; $\hat{\Omega} \subseteq \Omega$ be the safe territory where intermediate distribution centers can be located, m^2; *ρ(x)* be the function describing the distribution of resources at point $x \in \Omega$, resource units/m^2; N and M are the number of intermediate and final points; S is the total amount of resource in the territory of Ω, resource units; τ_i^r is the i-th center of the r-th stage, and the centers of the *1-st* stage will be considered intermediate; b_i^r is its capacity (capacity, the maximum amount of services that the respective centers can offer), *r=I,II* of the resource units; $c_i^I\left(x,\tau_i^I\right)$ is the cost of delivering a resource from the point $x \in \Omega$ to the center τ_i^I, UAH/unit of resource; $c_i^I\left(x,\tau_i^I\right)$ is the cost of transporting the resource from τ_i^I to τ_j^{II}, UAH/unit of resource; $c_{ij}^{II}\left(\tau_i^I,\tau_j^{II}\right)$ is the cost of organizing a transportation from center τ_i^I до τ_j^{II}, UAH/unit of resource; a_i^I is the cost of organizing an intermediate center at the point τ_i^I τ_j^{II}, UAH/unit of resource; a_j^{II} are the fixed organizational costs of the center τ_j^{II}, UAH/unit of resource; v_{ij} is the amount of resource transported from τ_i^I to τ_j^{II} (or in the opposite direction), unit of resource, $i=\overline{1,N}$; $j=\overline{1,M}$.

It is required to place the centers $\tau_1^I,\ldots,\tau_N^I$ from $\hat{\Omega}$, to divide the given region into zones $\Omega_{\sigma_l}, l=\overline{1,L}$, which cover customers that have the same k nearest service (intermediate distribution) centers and volumes of transportation $v=\left\{v_{11},\ldots,v_{NM}\right\}$, for which the total cost of transporting resources and the organizational expenses for opening and equipping the centers were minimized.

Here, $\sigma_l=\left\{p_1^l,p,\ldots,p_k^l\right\}$ is a set of indices of centers associated with the subset Ω_{σ_l}. The partitioning of the region Ω and the determination of resource flows in the logistics network must be carried out taking into account the capacities of all second-stage centers τ_j^{II}.

For the mathematical formulation of the problem, we introduce the following notation: $\hat{N} = \{1,2,\ldots,N\}$ – the set of all center indices; $M(\hat{N},k)$ – the set of all k-element subsets of the set $\hat{N}$, $|M(\hat{N},k)| = C_N^k = L; \sigma_l = \left\{p_1^l, p_2^l,\ldots, p_k^l\right\}, l = \overline{1,L}$, – elements of the set $M(\hat{N},k)$.

Definition 1. A collection of subsets $\left\{\Omega_{\sigma_1},\Omega_{\sigma_2},\ldots,\Omega_{\sigma_L}\right\}$ from $\Omega \subset E^2$ is called a partition of the k-th order of the set Ω to its subsets $\Omega_{\sigma_1},\Omega_{\sigma_2},\ldots,\Omega_{\sigma_L}$, if

$$\bigcup_{l=1}^{L}\Omega_{\sigma_l} = \Omega,\ mes\left(\Omega_{\sigma_i} \cap \Omega_{\sigma_m}\right) = 0;\ \sigma_i, \sigma_m \in M(\hat{N},k),\ i \neq m,\ i, m = \overline{1,L};$$

where mes(·) denotes the measure of a set. The subsets Ω_{σ_j} are called subsets of the k-th order of the set Ω.

Let $\Sigma_{\Omega}^{N,k}$ denote the class of all possible partitions of k-th order of Ω into its subsets $\Omega_{\sigma_1},\ldots,\Omega_{\sigma_L}$:

$$\Sigma_{\Omega}^{N,k} = \Big\{\bar{\omega} = \left\{\Omega_{\sigma_1},\ldots,\Omega_{\sigma_L}\right\} : \bigcup_{l=1}^{L}\Omega_{\sigma_l} = \Omega,$$
$$mes(\Omega_{\sigma_i} \cap \Omega_{\sigma_j}) = 0,\ \sigma_l,\sigma_j \in M(\hat{N},k), i \neq j,\ \ i, j = \overline{1,L}\Big\}.$$

The mathematical model of two-stage optimal distribution of material resources with the placement of intermediate service centers and the zoning of the emergency area is formulated as follows:

Problem 1:

$$F\left(\bar{\omega},\tau^I,\nu\right) \to min, \tag{1}$$

$$F\left(\bar{\omega},\tau^I,\nu\right) = \beta_1 \frac{1}{k}\sum_{l=1}^{L}\int_{\Omega_{\sigma_l}}\sum_{i\in\sigma_l}\left(c\left(x,\tau_i^I\right) + a_i^I\right)\rho\left(x\right)dx +$$
$$+\beta_2\sum_{i=1}^{N}\sum_{j=1}^{M}\left(c_{ij}^{II}\left(\tau_i^I,\tau_j^{II}\right) + a_j^{II}\right)\nu_{ij},$$

on condition of:

$$\sum_{\substack{l=1\\ l:\, i\in\sigma_l}}^{L}\int_{\Omega_{\sigma_l}}\gamma_i^l\rho\left(x\right)dx = \sum_{j=1}^{M}\nu_{ij},\quad i = \overline{1,N}, \tag{2}$$

$$\sum_{i=1}^{N} v_{ij} = b_j^{II}, \quad j = \overline{1,M}, \tag{3}$$

$$\varpi \in \Sigma_{\Omega}^{N,k}, \quad v \in R_{NM}^{+}, \quad \tau^I \in \hat{\Omega}^N. \tag{4}$$

Here $\tau^I = \{\tau_1^I,...,\tau_N^I\}$, $\tau_i^I \in \hat{\Omega}$; $x = (x^{(1)}, x^{(2)}) \in \Omega$; $c(x, \tau_i^I)$, $i = \overline{1,N}$ are bounded functions defined on $\Omega \times \Omega$ function. The function $\rho(x)$ is bounded and nonnegative on Ω; $a_j^{II}, a_i^I \geq 0, b_j^{II} > 0$ are given constants, $i = \overline{1,N}$, $j = \overline{1,M}$; $\beta_1, \beta_2 \geq 0$, $\beta_1^2 + \beta_2^2 \neq 0$ are given parameters that specify the priority of the terms and account for their normalization and dimensionlessness; $\Sigma_{\Omega}^{N,k}$ is the class of all possible multiplex partitions of Ω; R_{NM}^{+} and the space of ($N \times M$) nonnegative real matrices.

The coefficients γ_i^l determine the share of the service market that the center τ_i occupies in the territory Ω_{σ_l} among the objects $\{\tau_{j_1^l}, \tau_{j_2^l}, \ldots, \tau_{j_k^l}\}$ that serve this territory, such that for all $i = \overline{1,N}$, $l = \overline{1,L}$:

$$\gamma_{j_1^l}^l + \gamma_{j_2^l}^l + \ldots + \gamma_{j_k^l}^l = 1, \ 0 \leq \gamma_i^l \leq 1, \ i \in \sigma_l. \tag{5}$$

If we assume that the distribution of demand for services over the entire region Ω is proportional to the capacities of the first-stage centers b_i^I, so, it will be for all $l = \overline{1,L}$ and for all it will be $i = \overline{1,N}$, $i \in \sigma_l$, are given by the following expression: $\gamma_i^l = b_i^I / \sum\limits_{q:\, q \in \sigma_l} b_q^I$. If the demand is distributed evenly among the centers, then $\gamma_i^l = \dfrac{1}{k}$, it will be, for all *i* and *l*.

The problems (1) – (4) for any fixed vector $\tau^I \in \Omega^N$ are solvable if the following conditions are met:

1) $S = \int\limits_{\Omega} \rho(x) dx$;

2) the capacities of the centers at the first stage are equal to b_i^I, $i = \overline{1,N}$;

3) the following conditions are met:

$$0 \leq b_i^I, b_j^{II} \leq S, \quad i = \overline{1,N}, j = \overline{1,M}; \ \sum_{i=1}^{N} b_i^I = \sum_{j=1}^{M} b_j^{II} = S. \tag{6}$$

The validity of the statement can be easily shown by analogy with the proof of the lemma in [19].

The method of solving the Problem **A** assumes its representation in terms of characteristic vector-functions of a partition of the *k-th* order of the set Ω.

Definition 2. The characteristic vector-function of a partition $\overline{\omega}=\left\{\Omega_{\sigma_1},...,\Omega_{\sigma_l},\ldots,\Omega_{\sigma_L}\right\}$ of the set Ω of the *k-th* order is called the vector-function defined $\chi(\cdot)=\left(\chi_1(\cdot),\ldots,\chi_l(\cdot),\ldots,\chi_L(\cdot)\right)$, determined by the formula: a.e. for $x\in\Omega$

$$\chi_l(x)=\begin{cases}1,\ x\in\Omega_{\sigma_l},\\ 0, x\in\Omega\setminus\Omega_{\sigma_l},\ l=\overline{1,L}.\end{cases}$$

To write the Problems (1) – (4) in terms of characteristic partitioning functions, we use the vector function $\lambda(\cdot)$, which we assume to be unknown, and the *NL*-dimensional vector with coordinates

$$\lambda_i^l=\begin{cases}1, \quad \text{if } \ i\in\sigma_l,\\ 0 \text{ in the other case,}\end{cases} \quad i=\overline{1,N},\ l=\overline{1,L},$$

where $\sigma_l=\left\{j_1^l,...,j_k^l\right\}$ the set of indices of centers, associated with Ω_{σ_l}.

Obviously, $\sum_{j=1}^{N}\lambda_j^l=k$ for all $l=\overline{1,L}$. The vector $\lambda^l=(\lambda_1^l,\ldots,\lambda_N^l)$ defines the indicators of the indices of the centers of the set σ_l with **N** and, therefore, will be used as a template for the component $\chi_l(x)$ of the characteristic partitioning vector function.

The *problem A* is written in the following equivalent form.

Problem B:

$$\min_{(\chi(\cdot),\tau^I,\nu)\in\Gamma_1^k\times\hat{\Omega}^N\times R_{NM}^+} I\left(\chi(\cdot),\tau^I,\nu\right),$$

where

$$I\left(\chi(\cdot),\tau^I,\nu\right)=\frac{\beta_1}{k}\int_\Omega\sum_{l=1}^{L}\sum_{i=1}^{N}\left((c(x,\tau_i^I)+a_i^I)\lambda_i^l\right)\rho(x)\chi_l(x)dx+$$

$$+\beta_2\sum_{i=1}^{N}\sum_{j=1}^{M}\left(c_{ij}^{II}(\tau_i^I,\tau_j^{II})+a_j^{II}\right)\nu_{ij},$$

on conditions:

$$\int_\Omega\sum_{l=1}^{L}\gamma_i^l\lambda_i^l\chi_l(x)\rho(x)dx=\sum_{j=1}^{M}\nu_{ij},\quad i=\overline{1,N}, \tag{7}$$

$$\sum_{i=1}^{N}\nu_{ij}=b_j^{II},\quad j=\overline{1,M}, \tag{8}$$

$$\Gamma_1^k = \left\{ \chi(\cdot) : \chi_l(x) = 0 \vee 1,\ l = \overline{1, L}, \sum_{l=1}^{L} \chi_l(x) = 1 \text{ a.e. for } x \in \Omega \right\}.$$

3. Theoretical part

To solve the Problem **B**, its LP-relaxation is performed, where the components $\chi_l(\cdot)$, $l = \overline{1, L}$, are allowed to take values in the interval [0, 1].

Problem C:

$$\min_{(\chi(\cdot), \tau^I, \nu) \in \Gamma_1^k \times \hat{\Omega}^N \times R_{NM}^{+}} I\left(\chi(\cdot), \tau^I, \nu\right),$$

on conditions (7), (8), where

$$\Gamma_2^k = \left\{ \chi(\cdot) = \left(\chi_1(\cdot), \ldots, \chi_l(\cdot), \ldots, \chi_L(\cdot)\right) : 0 \le \chi_l(x) \le 1, \right.$$
$$\left. l = \overline{1, L};\ \sum\nolimits_{l=1}^{L} \chi_l(x) = 1 \text{ a.e. for } x \in \Omega \right\}.$$

To solve the Problem **C**, we make use of elements of duality theory. The Lagrange functional of this problem can be written in the following form:

$$W\left((\chi(\cdot), \tau^I, \nu), (\psi_0(\cdot), \psi, \eta)\right) = \int_{\Omega} \sum_{l=1}^{L} \sum_{i=1}^{N} \left(\frac{\beta_1}{k} \left(c\left(x, \tau_i^I\right) + a_i^I \right) + \gamma_i^l \psi_i \right) \lambda_i^l \rho(x) \chi_l(x) dx +$$
$$+ \sum_{i=1}^{N} \sum_{j=1}^{M} \left(\beta_2 (c_{ij}^{II}\left(\tau_i^I, \tau_j^{II}\right) + a_j^{II}) - \eta_j - \psi_i \right) v_{ij} + \sum_{j=1}^{M} \eta_j b_j^{II} + \int_{\Omega} \psi_0(x) \left(\sum_{l=1}^{L} \chi_l(x) - 1 \right) dx.$$

The functional $W\left((\chi(\cdot), \tau^I, \nu), (\psi_0(\cdot), \psi, \eta)\right)$ is defined on the Cartesian product $\left(\Lambda \times \hat{\Omega}^N \times R_{NM}^{+}\right) \times \left(\Phi \times R^N \times R^M\right)$, where

$$\Lambda = \left\{ \chi(\cdot) \in L_2^L(\Omega) : 0 \le \chi_l(x) \le 1\ \forall x \in \Omega, l = \overline{1, L} \right\};\ \Phi = \left\{ \psi_0 : \psi_0(\cdot) \in L_2(\Omega) \right\}.$$

The problem which is dual to the problem **C** is formulated as follows:

$$H\left(\psi_0(\cdot), \psi, \eta\right) \to \max_{\left(\psi_0(\cdot), \psi, \eta\right) \in \Phi \times R^N \times R^M}, \tag{9}$$
$$H\left(\psi_0(\cdot), \psi, \eta\right) = \min_{(\chi, \tau^I, \nu) \in \Lambda \times \hat{\Omega}^N \times R_{NM}^{+}} W\left((\chi(\cdot), \tau^I, \nu), (\psi_0(\cdot), \psi, \eta)\right).$$

Let all the parameters in *W,* except for $\chi(\cdot)$, be admissible and fixed. Then

$$\min_{\chi(\cdot)\in\Lambda} W\Big((\chi(\cdot),\tau^I,v),(\psi_0(\cdot),\psi,\eta)\Big)=$$

$$=\sum_{i=1}^{N}\sum_{j=1}^{M}(\beta_2(c_{ij}^{II}(\tau_i^I,\tau_j^{II})+a_j^{II})-\eta_j-\psi_i)v_{ij}+\sum_{j=1}^{M}\eta_j b_j^{II}-\int_{\Omega}\psi_0(x)dx+$$

$$+\int_{\Omega}\sum_{l=1}^{L}\min_{0\le\chi_l(x)\le1}\left[\sum_{i=1}^{N}\left(\frac{\beta_1}{k}(c(x,\tau_i^I)+a_i)+\gamma_i^l\psi_i\right)\lambda_i^l\rho(x)+\psi_0(x)\right]\chi_l(x)dx.$$

We shall further use the notation: $d_i(x,\tau_i^I)=\beta_1(c(x,\tau_i^I)+a_i^I)/k.$

The minimal value of the latter term is attained at the function $\hat{\chi}$, which components almost everywhere for x with Ω satisfy the conditions: for $l=\overline{1,L}$

$$\hat{\chi}_l(x)=\begin{cases}1, \quad \text{if } \sum_{i=1}^{N}(d_i(x,\tau_i^I)+\gamma_i^l\psi_i)\lambda_i^l\rho(x)+\psi_0(x)<0,\\ 0, \quad \text{if } \sum_{i=1}^{N}(d_i(x,\tau_i^I)+\gamma_i^l\psi_i)\lambda_i^l\rho(x)+\psi_0(x)>0,\\ \alpha\in[0,1], \text{ if } \sum_{i=1}^{N}(d_i(x,\tau_i^I)+\gamma_i^l\psi_i)\lambda_i^l\rho(x)+\psi_0(x)=0.\end{cases} \tag{10}$$

If in this inequality and in (10) $\psi_0(x)=\hat{\psi}_0(x)$, then the following equality holds:

$$\sum_{l=1}^{L}\chi_l(x)-1=0\,.$$

This implies that among the components of the vector $\hat{\chi}(x)$ in (10) there is only one unit component, let its number be *l* and then:

$$\hat{\chi}_l(x)=\begin{cases}1, \text{ if } \sum_{i=1}^{N}\Big(d_i(x,\tau_i^I)+\gamma_i^l\psi_i\Big)\lambda_i^l\rho(x)+\hat{\psi}_0(x)=\\ \quad =\min_{s=\overline{1,L}}\sum_{i=1}^{N}\Big(d_i(x,\tau_i^I)+\gamma_i^s\psi_i\Big)\lambda_i^s\rho(x)+\hat{\psi}_0(x),\\ 0 \text{ otherwise}, \qquad\qquad \forall l=\overline{1,L}.\end{cases} \tag{11}$$

By substituting (11) into the expression for W when $\psi_0(\cdot)=\hat{\psi}_0(\cdot)$, we obtain:

$$\min_{\chi(\cdot)\in\Lambda} W\Big((\chi(\cdot),\tau^I,v),(\hat{\psi}_0(\cdot),\psi,\eta)\Big)=$$

$$=\sum_{i=1}^{N}\sum_{j=1}^{M}(\beta_2(c_{ij}^{II}(\tau_i^I,\tau_j^{II})+a_j^{II})-\eta_j-\psi_i)v_{ij}+\sum_{j=1}^{M}\eta_j b_j^{II}+$$

$$+\int_{\Omega}\min_{s=1,L}\sum_{i=1}^{N}\Big(d_i(x,\tau_i^I)+\gamma_i^s\psi_i\Big)\lambda_i^s\rho(x)dx=\bar{W}\Big((\tau^I,v),(\psi,\eta)\Big).$$

Thus, the problem dual to problem **C,** takes the form:

$$\bar{H}(\psi,\eta)\to\max_{(\psi,\eta)\in R^N\times R^M},\tag{12}$$

$$\bar{H}(\psi,\eta)=\min_{(\tau^I,v)\in\hat{\Omega}^N\times R_{NM}^{+}}\bar{W}\Big((\tau^I,v),(\psi,\eta)\Big)$$

Now let us assume that all parameters except v are fixed in the problem (12). Then:

$$\min_{v\geq 0}\sum_{i=1}^{N}\sum_{j=1}^{M}(\beta_2(c_{ij}^{II}(\tau^I,\tau^{II})+a_j^{II})-\eta_j-\psi_i)v_{ij}=$$

$$=\begin{cases}0, \text{ if } \beta_2(c_{ij}^{II}(\tau_i^I,\tau_j^{II})+a_j^{II})-\eta_j-\psi_i\geq 0,\\ -\infty, \text{ if } \beta_2(c_{ij}^{II}(\tau_i^I,\tau_j^{II})+a_j^{II})-\eta_j-\psi_i<0.\end{cases}\tag{13}$$

The values of v_{ij}^{*} which achieve the minimum value for v_{ij} in the function $\bar{W}\Big((\tau^I,v),(\psi,\eta)\Big)$ satisfy the condition:

$$v_{ij}^{*}=\begin{cases}>0, \quad \text{if } \beta_2(c_{ij}^{II}(\tau_i^I,\tau_j^{II})+a_j^{II})=\psi_i+\eta_j,\\ 0, \quad \text{if } \beta_2(c_{ij}^{II}(\tau_i^I,\tau_j^{II})+a_j^{II})>\psi_i+\eta_j.\end{cases}$$

Each inequity from the following system

$$\beta_2(c_{ij}^{II}(\tau_i^I,\tau_j^{II})+a_j^{II})-\eta_j-\psi_i\geq 0 \text{ or}$$

$$\eta_j\leq\beta_2(c_{ij}^{II}(\tau_i^I,\tau_j^{II})+a_j^{II})-\psi_i,\forall i=\overline{1,N},j=\overline{1,M}.$$

Is satisfied, if

$$\eta_j=\min_{i=1,N}\Big(\beta_2(c_{ij}^{II}(\tau_i^I,\tau_j^{II})+a_j^{II})-\psi_i\Big),\forall\ j=\overline{1,M}.$$

Thus, by substituting such expressions into W, the problem (12) takes the form:

$$Q(\psi) \to \max_{\psi \in R^N}, \qquad Q(\psi) = \min_{\tau^I \in \hat{\Omega}^N} G\left(\tau^I, \psi\right), \tag{13}$$

where

$$G\left(\tau^I, \psi\right) = \int_{\Omega} \min_{s=\overline{1,L}} \sum_{i=1}^{N} \left(d_i(x, \tau_i^I) + \gamma_i^s \psi_i \right) \lambda_i^s \rho(x) dx +$$

$$+ \sum_{j=1}^{M} b_j^{II} \min_{i=\overline{1,N}} \left(\beta_2 (c_{ij}^{II}(\tau_i^I, \tau_j^{II}) + a_j^{II}) - \psi_i \right).$$

Substituting the expression for $d_i(x, \tau_i^I)$ and avoiding the use of index indicators that form the combination σ_s, $s = \overline{1,L}$, we can write the function $G(\tau^I, \psi)$ as follows:

$$G(\tau^I, \psi) = \int_{\Omega} \min_{\substack{\sigma_l \in \mathrm{M}(\underline{N_s}, k) \\ l=\overline{1,L}}} \sum_{i=1}^{N} \left(\frac{\beta_1}{k} (c(x, \tau_i^I) + a_i^I) + \gamma_i^l \psi_i \right) \rho(x) dx +$$

$$+ \sum_{j=1}^{M} b_j^{II} \min_{i=\overline{1,N}} \left(\beta_2 (c_{ij}^{II}(\tau_i^I, \tau_j^{II}) + a_j^{II}) - \psi_i \right).$$

Thus, taking into account the non-negativity of the function $\rho(x)$, the characteristic functions of the subsets $\Omega_{\sigma_l}^*, l = \overline{1,L}$, which form the optimal multiplex partition of Ω in the *problem B*, are given by the following formula**:** for almost everywhere for $x \in \Omega$

$$\hat{\chi}_l(x) = \begin{cases} 1, \text{ if } \sum_{i=1}^{N} \left(d_i(x, \hat{\tau}_i^I) + \gamma_i^l \hat{\psi}_i \right) \lambda_i^l = \min_{s=\overline{1,L}} \sum_{i=1}^{N} \left(d_i(x, \tau_i^I) + \gamma_i^s \hat{\psi}_i \right) \lambda_i^s, \\ \\ 0 \quad \text{otherwise}, \qquad \forall l = \overline{1,L}, \end{cases} \tag{14}$$

where $(\hat{\tau}_1, ..., \hat{\tau}_N, \ \hat{\psi}_1, ..., \hat{\psi}_N)$ is the solution of the problem (13). To identify the links between the centers of the first and second stages – the values of $\hat{v}_{ij}$ of resource flows between them, the following linear programming problem of the transport type should be solved:

$$\sum_{i=1}^{N} \sum_{j=1}^{M} \left(c_{ij}^{II}(\tau_i^I, \tau_j^{II}) + a_j^{II} \right) v_{ij} \to \min_{v \in R_{NM}^+}, \tag{15}$$

$$\sum_{j=1}^{M} v_{ij} = \int_{\Omega} \sum_{l=1}^{L} \gamma_i^l \lambda_i^l \hat{\chi}_l(x) \rho(x) dx, \quad i = \overline{1,N}, \tag{16}$$

$$\sum_{i=1}^{N} v_{ij} = b_j^{II}, \quad j = \overline{1, M}. \tag{17}$$

At the core of the numerical algorithm for solving the problem lies the following scheme:

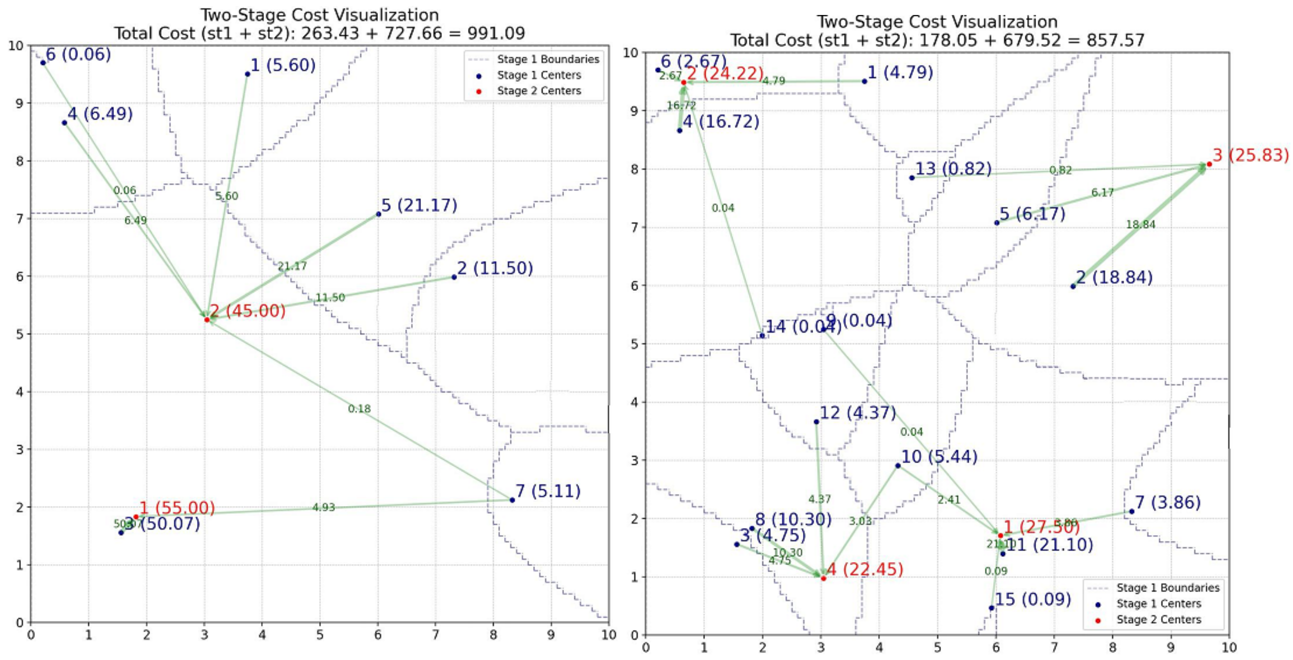
1. We solve the problem (13) using any method with the help of Shor's r-algorithm and determine the optimal location of the first-stage centers and the characteristic functions of the subsets that form the optimal multiplex partition of the set Ω. Thus, each first-stage center is assigned its service area, taking into account that the zones may overlap up to *k* times.
2. We calculate the capacities of the first-stage centers based on the defined service areas, considering the multiplicity of the partition.
3. We solve the problems (15)–(17) using the potentials method and determine the optimal values of the resource flows between the first- and second-stage centers.
4. We visualize the calculation results.

4. Results and discussion

To verify the correctness of the constructed models and the developed algorithms, a series of computational experiments was conducted. Firstly, particular cases of the formulated problems were solved, where the service areas are monotonically assigned to each intermediate center. In this case, the obtained results are identical to those presented in [17, 19]. Secondly, two-stage resource distribution problems were solved, where the centers are fixed but overlapping of their service areas is allowed. The results confirm the adequacy of the mathematical description of such processes using the developed model (1) – (4). Thus, Figure 1 presents a visualization of the optimal two-stage distribution of a total resource amount of 100 units, uniformly distributed over a square territory, among N intermediate centers (collecting the resource) and M destination centers. The amount of collected resource and the amount directed to the second-stage centers are indicated. The capacity of each second-stage center, along with its number, is shown in red. The coordinates of the intermediate centers are predetermined. The zones are defined for monopoly servicing (k = 1). Next to the number of each first-stage center, the amount of collected resource is given in parentheses (blue color). The figure captions indicate the values of the components of the objective functional: costs at the first and second stages of distribution and delivery of the resource, as well as their sum.

As can be seen, the capacities of certain intermediate centers, as well as the resource flows between the first- and second-stage centers, may sometimes turn out to be too small. Their practical implementation may be associated with certain difficulties. For instance, servicing such centers and flows is not rational from the standpoint of using transportation means or involving human resources. To prevent such situations, two consecutive procedures for improving the scheme of two-stage rational resource distribution are proposed.

The first procedure is the exclusion from consideration of intermediate centers which, according to the optimal partitioning of the territory, turn out to be lowcapacity. Their resources are then redistributed among the remaining centers, for which the formulated problem is solved again. The second procedure is developed to avoid small resource flows between the first- and second-stage centers. It is assumed that the second-stage centers can accept somewhat more resources than initially specified. Their capacities are artificially increased by a small value. This increment is determined by a lower bound on the amount of resource flow between the first- and second-stage centers (i.e., the minimum amount of resource transportation worth performing from the perspective of process organization).

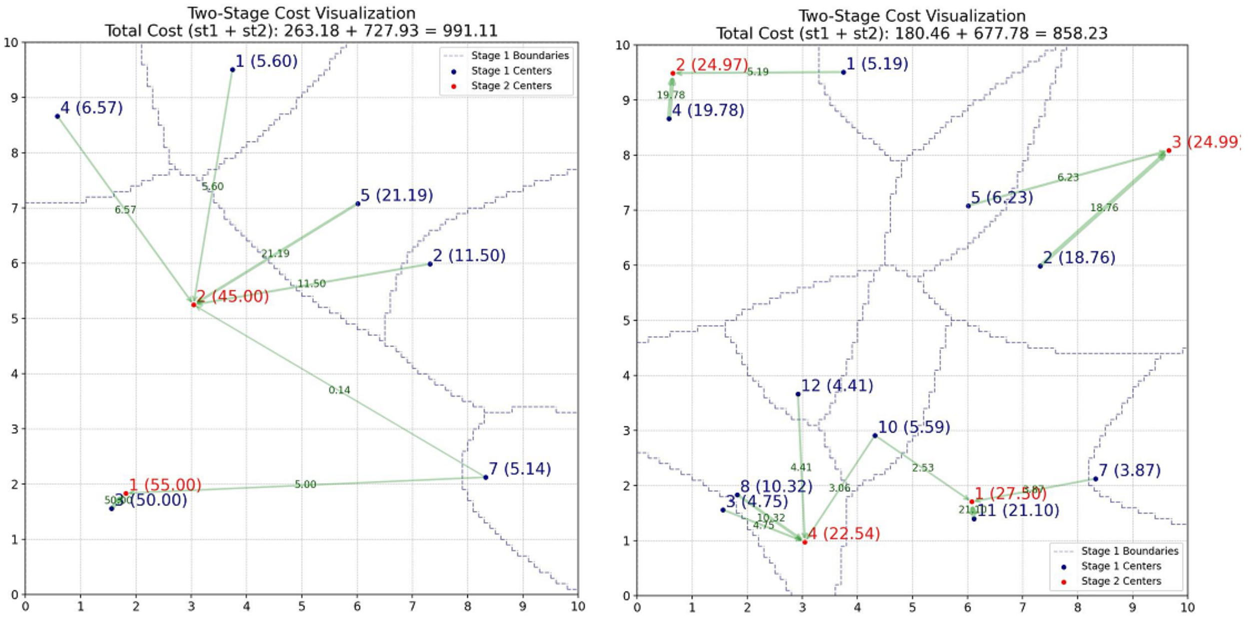


a) $N = 7$, $M = 2$ b) $N = 15$, $M = 4$

Figure 1 – Solution of the resource flow optimization problem with N fixed intermediate centers and M second-stage centers

A fictitious intermediate center is introduced with a capacity equal to the total increment of the capacities of the second-stage centers, and with a transportation cost that significantly exceeds the total cost of all actual shipments. The extended transportation problem is then solved. Figure 2 shows the result of applying the described rationalization procedures to the two-stage resource distribution process for the optimal solutions presented in Figure 1.

In the optimal partition in Figure 1a, the service zone of the 6th center turned out to be very small, with the amount of collected resource equal to 0.06 units. After excluding this center from consideration, the obtained optimal partition is shown in Figure 2a. At the same time, the value of the objective functional is increased by only 0.002% – from 991.09 to 991.11. In Figure 2b, a rational partition of the territory into service zones is shown for ten out of the fifteen centers presented in Figure 1b – centers 6, 9, 13, 14, and 15 were found to be irrational for use and were excluded from consideration. Given a 10% increment of the capacities of the second-stage centers, eleven active links between the centers of the two stages were obtained.



a) $N = 7$, $M = 2$ b) $N = 15$, $M = 4$

Figure 2 – Result of rationalizing the scheme of connections between first- and second-stage centers

Compared with the connections presented in Figure 1b, in Figure 2b there are no flows of 0.04, 0.09, or 0.89 units of resource. The total costs due to the redistribution of resource flows increased only slightly – from 857.57 to 858.23 conventional units, which is almost 0.08%. Thus, the calculation results confirm the expediency of applying rationalization procedures to the optimal solutions of two-stage location–allocation problems.

5. Conclusion

Mathematical models, methods, and algorithms have been developed to determine resource flows in multi-level transport and logistics systems where a resource is continuously distributed over a given territory, and intermediate logistics centers are located within that territory. It is assumed that each consumer of the service is assigned to several of the nearest intermediate centers to increase the reliability of service. The developed mathematical framework is a symbiosis of models and methods of two-stage location–allocation and multiplex partitioning of continuous sets. A numerical solution scheme for the formulated problems has been presented. Thus, optimization models similar to (1)–(4) have the advantage of providing exact solutions for the problems with clearly defined objectives and constraints. The proposed approach makes it possible to comprehensively solve the problems of locating facilities in emergency logistics systems and distributing resources during their transportation from the sources to the final consumers, thereby strengthening the synergistic effect of managerial decision-making. The presented mathematical models are suitable for describing evacuation processes with the organization of collection, intermediate, and reception points; for multi-stage processes of distributing and delivering essential goods from existing warehouses through distribution centers to areas affected by man-made emergencies; and for other logistics processes.

However, to ensure controllability, these models often simplify the complexity of scenarios by quantitatively estimating uncertainty and relying on strong assumptions about human behavior. These limitations can be overcome using machine learning and simulation methods, which provide enhanced realism in modeling disaster scenarios [15]. The most important directions for future research include:

- rapid adaptation to environmental changes based on online learning and reinforcement learning (RL);
- accurate forecasting under complex conditions with additional optimization frameworks;
- multi-objective optimization for balancing conflicting priorities and achieving fair outcomes;
- application of probabilistic risk analysis and stochastic optimization methods to quantify risk and develop robust mitigation strategies.

Conflict of interest

Authors state no conflict of interest.

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УДОСКОНАЛЕННЯ МАТЕМАТИЧНИХ МОДЕЛЕЙ ДВОЕТАПНИХ ПРОЦЕСІВ РОЗПОДІЛУ РЕСУРСІВ В СИСТЕМАХ ЕКСТРЕНОЇ ЛОГІСТИКИ

Блюсс Б., Дзюба С., Коряшкіна Л., Лубенець Д.

Анотація. У статті розглядається проблема оптимізації розміщення об’єктів і розподілу матеріальних ресурсів у контексті логістики надзвичайних ситуацій. Актуальність дослідження зумовлена зростанням частоти природних та техногенних катастроф, що вимагає ефективного планування заходів реагування задля зменшення втрат і загроз для життя населення. Основна увага приділяється удосконаленню математичних моделей і методів, які дозволяють підвищити ефективність логістичних операцій.

Запропоновано новий підхід щодо математичного моделювання багатоетапних логістичних процесів, який поєднує ідеї оптимального розміщення проміжних розподільчих центрів та мультиплексного зонування територій. Існуючі моделі задач розташування логістичних центрів з двоетапним розподілом ресурсу удосконалено завдяки закріпленню кожного кінцевого споживача за кількома найближчими проміжними розподільчими центрами, що підвищує надійність надання послуги потребуючим. Центрами, що розміщуються, можуть бути або пункти первинного збору населення на випадок його евакуації з території надзвичайної ситуації, або пункти розподілу і видачі предметів першої необхідності, які надходять зі складів державного резерву або інших хабів і розподіляються між проміжними центрами для доставки мешканцям постраждалого регіону. Розроблений математичний апарат дозволяє визначати оптимальну кількість, місткість і місця розташування проміжних розподільчих центрів, а також раціонально організовувати логістичні процеси та ефективно розподіляти ресурси між усіма учасниками логістичного ланцюга.

Математичне забезпечення сформульованих задач ґрунтується на теорії неперервних задач оптимального розбиття множин, теорії двоїстості, методах лінійного програмування транспортного типу, а також сучасних алгоритмах недиференційованої оптимізації. Запропоновано чисельну схему розв’язання задач оптимального розподілу потоків у багаторівневих транспортно-логістичних мережах.

Представлені результати мають прикладне значення для планування логістичних операцій у надзвичайних ситуаціях, зокрема для ефективного розміщення пунктів медичної та гуманітарної допомоги, розподілу ресурсів і евакуації населення. Запропонований підхід дозволяє реалізувати комплексну систему підтримки прийняття управлінських рішень у сфері кризового реагування, може бути використаний для вирішення широкого кола стратегічних завдань у виробничій, соціальній та економічній сферах, забезпечуючи ефективну підтримку прийняття управлінських рішень у складних логістичних системах.

Ключові слова: оптимізація, транспортно-логістична система, багатоетапний розподіл, мультиплексне розбиття континуальної множини, розміщення об’єктів, системний підхід.